

Задача 1 (2 балла)

Для заданных функций φ и ψ :

- а) построить графики функций φ и ψ ;
- б) вычислить свертку $\varphi * \psi$ функций φ и ψ ;
- в) построить график свертки $\varphi * \psi$.

Функция φ задана формулой, график функции ψ — ломаная, соединяющая точки $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$ (вне отрезка $[x_1, x_4]$ функция равна нулю).

Вар.	$\varphi(x)$	A	B	C	D	Вар.	$\varphi(x)$	A	B	C	D
1	$\text{rect} x$	(-2,1)	(-1,-1)	(1,-1)	(2,1)	16	$-\text{rect} x$	(-2,0)	(-1,2)	(1,2)	(2,0)
2	$\text{rect} \frac{x}{2}$	(-2,1)	(0,-1)	(1,-1)	(3,1)	17	$-\text{rect} \frac{x}{2}$	(-2,0)	(0,2)	(1,2)	(3,0)
3	$\text{rect} x$	(-2,0)	(-1,2)	(1,2)	(2,0)	18	$-\text{rect} x$	(-2,-2)	(-1,0)	(1,0)	(2,2)
4	$\text{rect} \frac{x}{2}$	(-2,0)	(0,2)	(1,2)	(3,0)	19	$-\text{rect} x$	(-2,2)	(-1,0)	(1,0)	(2,-2)
5	$\text{rect} x$	(-2,-2)	(-1,0)	(1,0)	(2,2)	20	$-\text{rect} x$	(0,0)	(1,-2)	(3,-2)	(4,0)
6	$\text{rect} x$	(-2,2)	(-1,0)	(1,0)	(2,-2)	21	$-\text{rect} \frac{x}{2}$	(-2,1)	(-1,-1)	(1,-1)	(2,1)
7	$\text{rect} x$	(0,0)	(1,-2)	(3,-2)	(4,0)	22	$-\text{rect} x$	(0,2)	(2,0)	(3,0)	(5,2)
8	$\text{rect} \frac{x}{2}$	(-2,1)	(-1,-1)	(1,-1)	(2,1)	23	$-\text{rect} x$	(0,-1)	(1,1)	(3,1)	(4,-1)
9	$\text{rect} x$	(0,2)	(2,0)	(3,0)	(5,2)	24	$-\text{rect} \frac{x}{2}$	(0,2)	(2,0)	(3,0)	(5,2)
10	$\text{rect} x$	(0,-1)	(1,1)	(3,1)	(4,-1)	25	$-\text{rect} \frac{x}{2}$	(0,-1)	(1,1)	(3,1)	(4,-1)
11	$\text{rect} \frac{x}{2}$	(0,2)	(2,0)	(3,0)	(5,2)	26	$-\text{rect}(x-1)$	(-2,1)	(-1,-1)	(1,-1)	(2,1)
12	$\text{rect} \frac{x}{2}$	(0,-1)	(1,1)	(3,1)	(4,-1)	27	$-\text{rect} x$	(-2,1)	(-1,-1)	(1,-1)	(2,1)
13	$\text{rect}(x-1)$	(-2,1)	(-1,-1)	(1,-1)	(2,1)	28	$-\text{rect} \frac{x}{2}$	(-2,1)	(0,-1)	(1,-1)	(3,1)
14	$\text{rect}(x-1)$	(-2,1)	(-1,-1)	(1,-1)	(2,1)	29	$-\text{rect} x$	(-2,1)	(-1,-1)	(1,-1)	(2,1)
15	$\text{rect} x$	(-2,1)	(0,-1)	(1,-1)	(3,1)	30	$-\text{rect} \frac{x}{2}$	(-2,1)	(0,-1)	(1,-1)	(3,1)

Примечание 1. Если $f(x) = 0$ при $x < a$, а $g(x) = 0$ при $x < b$, $a < b$, то

$$(f * g)(x) = \eta(x - a - b) \int_b^{x-a} f(x - \xi) g(\xi) d\xi.$$

Примечание 2.

$$\text{rect} x = \eta\left(\frac{1}{2} - |x|\right) = \begin{cases} 1, & -\frac{1}{2} < x < \frac{1}{2}; \\ 0, & x < -\frac{1}{2} \text{ или } x > \frac{1}{2}. \end{cases}$$

Задача 2 (2 балла)

Определите тип дифференциального уравнения, приведите его к каноническому виду, запишите общее решение, найдите решение задачи Коши.

1. $u_{xx} - 2x u_{xy} + x^2 u_{yy} - u_y = 0; \quad u(0, y) = y^2, u_x(0, y) = y.$
2. $u_{xx} + 2(\sin x)u_{xy} + (\sin^2 x)u_{yy} + (\cos x)u_y = 0; \quad u(0, y) = y^2, u_x(0, y) = y^3.$
3. $y^4 u_{xx} + 2y^2 u_{xy} + u_{yy} - \frac{2}{y} u_y = 0; \quad u(x, 1) = \frac{x^3}{3}, u_y(x, 1) = 2x.$
4. $4y^2 u_{xx} + 2(1 - y^2)u_{xy} - u_{yy} - \frac{4y}{1 + y^2} u_x + \frac{2y}{1 + y^2} u_y = 0; \quad u(x, 1) = x, u_y(x, 1) = 0.$
5. $y^2 u_{xx} - 2y u_{xy} + u_{yy} - u_x = 0; \quad u(x, 1) = \left(x + \frac{1}{2}\right)^2, u_y(x, 1) = 0.$
6. $u_{xx} - 2(\cos x)u_{xy} - (3 + \sin^2 x)u_{yy} + (\sin x)u_y = 0; \quad u(0, y) = 0, u_x(0, y) = y^2.$
7. $y^2 u_{xx} - 2y u_{xy} + u_{yy} + u_x - \frac{2}{y} u_y = 0; \quad u(x, 1) = x^2, u_y(x, 1) = x.$
8. $u_{xx} - 2(\sin x)u_{xy} - (\cos^2 x)u_{yy} - u_x + (\sin x - \cos x - 1)u_y = 0; \quad u(0, y) = 3y, u_x(0, y) = 5.$
9. $u_{xx} + 2(\sin x)u_{xy} + (\sin^2 x)u_{yy} - u_x - (\sin x - \cos x)u_y = 0; \quad u(0, y) = y^2, u_x(0, y) = y.$
10. $u_{xx} + 2(\cos x)u_{xy} - (\sin^2 x)u_{yy} - (\sin x)u_y = 0; \quad u(0, y) = y^2, u_x(0, y) = 1.$
11. $u_{xx} + 2x^2 u_{xy} + x^4 u_{yy} + u_x + (x^2 + 2x)u_y = 0; \quad u(0, y) = y^2, u_x(0, y) = y.$
12. $4y^3 u_{xx} - y u_{yy} + 2y^3 u_x + (1 + y^2)u_y = 0; \quad u(x, 1) = x^2, u_y(x, 1) = 0.$
13. $9y^5 u_{xx} - y u_{yy} + 18y^5 u_x + (2 - 6y^3)u_y = 0; \quad u(x, 1) = 0, u_y(x, 1) = x.$
14. $u_{xx} - 2(\sin x)u_{xy} - (\cos^2 x)u_{yy} - 2u_x + (2 \sin x + 2 - 2 \cos x)u_y = 0; \quad u(0, y) = \frac{y^2}{2}, u_x(0, y) = 1.$
15. $-x u_{xx} + 4x^3 u_{yy} + (1 - 4x^2)u_x + 8x^3 u_y = 0; \quad u(1, y) = y, u_x(1, y) = 3.$
16. $y^2 u_{xx} - 2y u_{xy} + u_{yy} - \frac{1}{y} u_y = 0; \quad u(x, 1) = x, u_y(x, 1) = x^2.$
17. $u_{xx} - 2(\sin x)u_{xy} - (\cos^2 x)u_{yy} + 2u_x - (2 + \cos x + 2 \sin x)u_y = 0; \quad u(0, y) = 2y, u_x(0, y) = 1.$
18. $u_{xx} + 2x^2 u_{xy} + x^4 u_{yy} + 2x u_y = 0; \quad u(0, y) = y, u_x(0, y) = y^2.$
19. $u_{xx} + 2x^2 u_{xy} + x^4 u_{yy} - u_x + (2x - x^2)u_y = 0; \quad u(0, y) = \sin y, u_x(0, y) = y.$
20. $y^2 u_{xx} + 2y u_{xy} + u_{yy} + (1 - y)u_x - u_y = 0; \quad u(x, 0) = x^2, u_y(x, 0) = x.$
21. $-x u_{xx} + 9x^5 u_{yy} + (2 - 6x^3)u_x + 18x^5 u_y = 0; \quad u(1, y) = 0, u_x(1, y) = y.$
22. $y^2 u_{xx} + 2y u_{xy} + u_{yy} + (1 + y)u_x + u_y = 0; \quad u(x, 0) = -x, u_y(x, 0) = \sin x.$
23. $u_{xx} - 2(\sin x)u_{xy} - (\cos^2 x)u_{yy} + u_x + (1 - \cos x - \sin x)u_y = 0; \quad u(0, y) = y, u_x(0, y) = 0.$
24. $y^2 u_{xx} + 2y u_{xy} + u_{yy} + u_x = 0; \quad u(x, 0) = x^3, u_y(x, 0) = -x.$
25. $(\sin^2 y)u_{xx} + 2(\cos y)u_{xy} - u_{yy} - (\sin y)u_x = 0; \quad u(x, 0) = x^2, u_y(x, 0) = 1.$
26. $u_{xx} - 2x u_{xy} + x^2 u_{yy} - u_x + (x - 1)u_y = 0; \quad u(0, y) = y, u_x(0, y) = y^2.$
27. $(3 + \sin^2 y)u_{xx} - 2(\cos y)u_{xy} - u_{yy} + (\sin y)u_x = 0; \quad u(x, 0) = x, u_y(x, 0) = x^2.$
28. $9y^5 u_{xx} - y u_{yy} + 6y^5 u_x + (2 + 2y^3)u_y = 0; \quad u(x, 1) = 2x, u_y(x, 1) = 0.$
29. $(\cos^2 y)u_{xx} - 2(\sin y)u_{xy} - u_{yy} + (1 - \cos y + \sin y)u_x + u_y = 0; \quad u(x, 0) = x^2, u_y(x, 0) = 0.$
30. $-x u_{xx} + 4x^3 u_{yy} + (1 + x^3)u_x + 2x^3 u_y = 0; \quad u(1, y) = y^2, u_x(1, y) = 0.$

Задача 3 (2 балла)

Найти поток векторного поля \mathbf{F} через поверхность S , ограничивающую конечный объем. Результат проверить по формуле Остроградского — Гаусса.

1. $\mathbf{F}(x, y, z) = (x + z)z\mathbf{i} + x^2\mathbf{j} + (x - 1)(x - z)\mathbf{k}$,
 $S: x + z = 1, x = 0, y = 0, z = 0, x + y = 1.$
2. $\mathbf{F}(x, y, z) = (x + z)z\mathbf{i} - x^2\mathbf{j} + (x - 1)(x + z)\mathbf{k}$,
 $S: x + z = 1, x = 0, y = 0, z = 0, x + y = 1.$
3. $\mathbf{F}(x, y, z) = (x + z)y\mathbf{i} + (2x^2 - y^2)\mathbf{j} + (1 - x)(x + z)\mathbf{k}$,
 $S: x + z = 1, y = 0, z = 0, y = x.$
4. $\mathbf{F}(x, y, z) = (x + y)(y - 1)\mathbf{i} + x(y + x)\mathbf{j} + (z - xy)\mathbf{k}$,
 $S: x + y = 1, y = 0, z = 1, x + z = 1.$
5. $\mathbf{F}(x, y, z) = 2x^2y^3\mathbf{i} + (y + z)z\mathbf{j} + (1 - x)(y + z)\mathbf{k}$,
 $S: y + z = 1, x = 1, y = 0, z = 0, z = x.$
6. $\mathbf{F}(x, y, z) = (x^2 + 2y^2)\mathbf{i} + x(y + z)\mathbf{j} + (x - 1)(y + z)\mathbf{k}$,
 $S: y + z = 1, x = 0, z = 0, y = x.$
7. $\mathbf{F}(x, y, z) = (x + y)(1 - y)\mathbf{i} + x(x + y)\mathbf{j} + (y^3 - x^3)\mathbf{k}$,
 $S: x + y = 1, x = 0, z = 0, z = y.$
8. $\mathbf{F}(x, y, z) = 2x^2z\mathbf{i} + (y + z)^2\mathbf{j} + x(y + z)\mathbf{k}$,
 $S: y + z = 1, x = 0, y = 0, x + y = 1, z = 0.$
9. $\mathbf{F}(x, y, z) = (x + y)z\mathbf{i} + (1 - x)(x + y)\mathbf{j} + 2x^2y^3\mathbf{k}$,
 $S: x + y = 1, y = 0, z = 0, z = x.$
10. $\mathbf{F}(x, y, z) = (x^2 - 2y^3)\mathbf{i} + x(y + z)\mathbf{j} + y(y + z)\mathbf{k}$,
 $S: y + z = 1, x = 0, y = 0, x + y = 1, z = 0.$
11. $\mathbf{F}(x, y, z) = (x + y)z\mathbf{i} + (1 - x)(x + y)\mathbf{j} + 2x^2y^3\mathbf{k}$,
 $S: x + y = 1, y = 0, z = 0, z = x.$
12. $\mathbf{F}(x, y, z) = (x^2 + 2y^2)\mathbf{i} + x(y + z)\mathbf{j} + (x - 1)(y + z)\mathbf{k}$,
 $S: y + z = 1, x = 0, y = 0, x + y = 1, z = 0.$
13. $\mathbf{F}(x, y, z) = (y^2\mathbf{i} + xz\mathbf{j} + (x + 1)(y + z)\mathbf{k})$,
 $S: x + z = 1, x = 0, y = 0, z = 0, x + y = 1.$
14. $\mathbf{F}(x, y, z) = (y + z)y\mathbf{i} + x^2\mathbf{j} + (y^3 - x^3)\mathbf{k}$,
 $S: x + y = 1, x = 0, z = 0, z = y.$
15. $\mathbf{F}(x, y, z) = (x + z)y\mathbf{i} + z(x + y)\mathbf{j} + 2x^2y\mathbf{k}$,
 $S: x + y = 1, y = 0, z = 0, z = x.$
16. $\mathbf{F}(x, y, z) = 2x^2y\mathbf{i} + xz^2\mathbf{j} - 2xyz\mathbf{k}$,
 $S: z = x^2 + y^2, z = 1, y = 0 (y \geq 0).$
17. $\mathbf{F}(x, y, z) = -x^2y\mathbf{i} + xy^2\mathbf{j} + x^2z\mathbf{k}$,
 $S: z^2 = x^2 + y^2, z = 2, x = 0, y = 0 (x \geq 0, y \geq 0).$
18. $\mathbf{F}(x, y, z) = xz\mathbf{i} - yz\mathbf{j} + z^2\mathbf{k}$,
 $S: x^2 + y^2 + z^2 = 4, z = 1 (z \geq 1).$
19. $\mathbf{F}(x, y, z) = x^2y\mathbf{i} - xy^2\mathbf{j} + xz^2\mathbf{k}$,
 $S: z = x^2 + y^2, z = 2, y = 0 (y \geq 0).$

20. $\mathbf{F}(x, y, z) = x^2y\mathbf{i} - xy^2\mathbf{j} + y^2z\mathbf{k}$,
 $S: z^2 = x^2 + y^2, z = 1, x = 0, y = 0 (x \geq 0, y \geq 0)$.
21. $\mathbf{F}(x, y, z) = x^2\mathbf{i} - y^2\mathbf{j} + z^2\mathbf{k}$,
 $S: 4 = x^2 + y^2, z = 1, z = 5$.
22. $\mathbf{F}(x, y, z) = xy^2\mathbf{i} + yz^2\mathbf{j} - 2xy\mathbf{k}$,
 $S: z = x^2 + y^2, z = 2, x = 0 (x \geq 0)$.
23. $\mathbf{F}(x, y, z) = xy\mathbf{i} + y^2z\mathbf{j} + xyz\mathbf{k}$,
 $S: y^2 = x^2 + z^2, y = 2, x = 0, z = 0 (x \geq 0, z \geq 0)$.
24. $\mathbf{F}(x, y, z) = yz\mathbf{i} - xz\mathbf{j} + x^2\mathbf{k}$,
 $S: x^2 + y^2 + z^2 = 4, x = 1 (x \geq 1)$.
25. $\mathbf{F}(x, y, z) = xz^2\mathbf{i} - y^2z\mathbf{j} + xy^2\mathbf{k}$,
 $S: x = y^2 + z^2, x = 2, z = 0 (z \geq 0)$.
26. $\mathbf{F}(x, y, z) = xy^2\mathbf{i} - x^2y\mathbf{j} + x^2z\mathbf{k}$,
 $S: z^2 = x^2 + y^2, z = 2, x = 0, y = x (x, y \geq 0)$.
27. $\mathbf{F}(x, y, z) = 2y^2\mathbf{i} + x^2z\mathbf{j} - 2yz\mathbf{k}$,
 $S: z = x^2 + y^2, z = 2, y = 0, y = x (x, y \geq 0)$.
28. $\mathbf{F}(x, y, z) = xy\mathbf{i} + xz\mathbf{j} + y^2z\mathbf{k}$,
 $S: x^2 = y^2 + z^2, x = 1, y = x, y = -x (y \geq 0)$.
29. $\mathbf{F}(x, y, z) = yz^2\mathbf{i} - x^2z\mathbf{j} + xy^2\mathbf{k}$,
 $S: z = x^2 + y^2, z = 1, y = 0, y = x (x \geq 0)$.
30. $\mathbf{F}(x, y, z) = xy\mathbf{i} + x^2\mathbf{j} + y^2z\mathbf{k}$,
 $S: y^2 = x^2 + z^2, y = 2, x = 0, x = z (x \geq 0)$.

Задача 4 (2 балла)

Функция f представляет собой ломаную, соединяющую точки A, B, C, D, E, F . Левее точки A и правее точки F она равна нулю. Функция g задана формулой: $g(x) = \text{rect } x$. Найти:

- а) преобразование Фурье функции f , используя преобразования Фурье стандартных функций $\text{rect } x$ и $\Lambda(x)$, а также свойства преобразования Фурье;
 б) преобразование Фурье свертки $f * g$.

Вар.	A	B	C	D	E	F
1	(0, 0)	(1, -2)	(2, -1)	(3, -1)	(4, -2)	(5, 0)
2	(0, 1)	(1, 3)	(2, 2)	(3, 3)	(4, 1)	(6, 1)
3	(0, 2)	(1, 1)	(2, 1)	(4, 3)	(5, 3)	(6, 2)
4	(0, 0)	(1, 2)	(2, 3)	(3, 3)	(4, 2)	(5, 0)
5	(0, 2)	(2, 4)	(3, 4)	(6, 1)	(7, 1)	(8, 2)
6	(0, 0)	(1, 2)	(2, 2)	(3, 1)	(4, 2)	(5, 0)
7	(0, 0)	(2, -2)	(3, -1)	(4, -2)	(5, -2)	(7, 0)
8	(0, 1)	(3, 4)	(4, 3)	(5, 3)	(7, 1)	(9, 1)
9	(0, 0)	(3, 3)	(4, 2)	(5, 2)	(6, 3)	(9, 0)
10	(0, 1)	(3, 4)	(4, 3)	(5, 3)	(8, 0)	(9, 1)
11	(0, 3)	(1, 1)	(2, 0)	(3, 0)	(4, 1)	(5, 3)
12	(0, 2)	(1, 0)	(2, 1)	(3, 0)	(4, 2)	(7, 2)
13	(0, 1)	(1, 1)	(3, 3)	(4, 3)	(5, 4)	(8, 1)
14	(1, 1)	(3, 3)	(4, 2)	(5, 3)	(6, 3)	(8, 1)
15	(0, -2)	(1, -3)	(2, -3)	(5, 0)	(7, -2)	(9, -2)
16	(0, 1)	(1, 1)	(2, 2)	(5, 2)	(6, 1)	(7, 1)
17	(0, 0)	(1, 2)	(2, 3)	(3, 2)	(4, 2)	(5, 0)
18	(0, 1)	(2, 3)	(3, 3)	(4, 4)	(8, 0)	(9, 1)
19	(0, -1)	(1, -1)	(2, 0)	(5, -3)	(7, -3)	(9, -1)
20	(0, -1)	(2, 1)	(3, 0)	(5, 2)	(8, -1)	(9, -1)
21	(0, 3)	(2, 3)	(4, 1)	(5, 2)	(7, 0)	(10, 3)
22	(0, 0)	(2, 2)	(4, 2)	(5, 1)	(6, 1)	(7, 0)
23	(0, 0)	(1, -1)	(2, -1)	(3, -2)	(5, -2)	(7, 0)
24	(-2, 3)	(0, 5)	(4, 1)	(5, 2)	(7, 0)	(10, 3)
25	(2, 3)	(4, 1)	(5, 2)	(7, 0)	(12, 5)	(14, 3)
26	(1, 1)	(3, 3)	(4, 3)	(5, 2)	(6, 3)	(8, 1)
27	(0, 1)	(1, 0)	(4, 3)	(5, 3)	(6, 4)	(9, 1)
28	(0, 2)	(1, 1)	(4, 1)	(6, 3)	(7, 3)	(8, 2)
29	(0, 1)	(1, 1)	(2, 3)	(3, 2)	(4, 1)	(5, 1)
30	(0, 1)	(1, 1)	(2, 3)	(3, 3)	(5, 1)	(6, 1)

Примечание 1.

$$\mathcal{F}[\text{rect}](\omega) = \text{sinc } \omega; \quad \mathcal{F}[\Lambda](\omega) = \text{sinc}^2 \frac{\omega}{2}.$$

Примечание 2.

$$\text{rect } x = \eta\left(\frac{1}{2} - |x|\right); \quad \Lambda(x) = (1 - |x|)(\eta(x + 1) - \eta(x - 1)); \quad \text{sinc } x = \frac{\sin x}{x}.$$